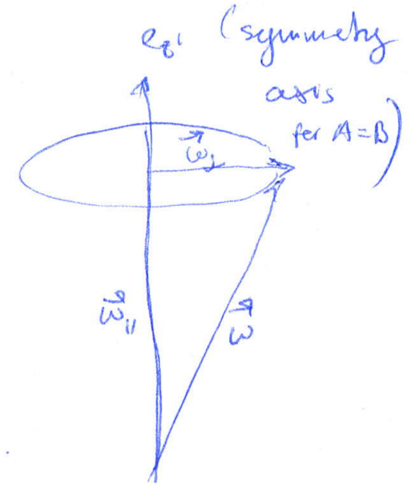
Recall from last time

- triaxial : intermediate axis theorem
- symmetric top

$\hookrightarrow$ $\ddot{\omega}_{x'} + \Omega^2 \omega_{x'} = 0$

$\ddot{\omega}_{y'} + \Omega^2 \omega_{y'} = 0$, $\Omega^2 = \frac{(A-C)^2}{A^2}\, \omega_{\parallel}^2$



$\Rightarrow$ $\omega_{x'}(t) = \omega_{\perp} \sin(\Omega t + \alpha)$

$\omega_{y'}(t) = \omega_{\perp} \cos(\Omega t + \alpha)$

- $\omega_{\perp}$ is const.
- $\omega = \sqrt{\omega_{\parallel}^2 + \omega_{\perp}^2} =$ const.
- $\vec{\omega}_{\perp}(t) = (\omega_{x'}(t), \omega_{y'}(t))$ is a circle in BF $x'y'$ plane.

$$(\vec{L})_{BF} = \begin{pmatrix} A\omega_{\perp} \sin(\Omega t + \alpha) \\ A\omega_{\perp} \cos(\Omega t + \alpha) \\ C\omega_{\parallel} \end{pmatrix} \neq \text{const.}!$$

$\Rightarrow$ $\vec{L}'$ rotates in body-fixed frame but $\vec{L} =$ const. in lab frame (no torques).

Assume $\vec{L} = L\vec{e}_z = \begin{pmatrix} L\sin\theta \sin\psi \\ L\sin\theta\cos\psi \\ L\cos\theta \end{pmatrix}$ (xz convention for Euler angles)

$(\vec{L})_{BF} = (\vec{L}_{BF})$ $\Rightarrow$

$L\sin\theta\sin\psi = A\omega_{\perp}\sin(\Omega t+\alpha) = A\dot\phi\sin\theta\sin\psi + A\dot\theta\cos\psi$

$L\sin\theta\cos\psi = A\omega_{\perp}\cos(\Omega t+\alpha) = A\dot\phi\sin\theta\cos\psi - A\dot\theta\sin\psi$

$L\cos\theta = C\omega_{\parallel} = C\dot\phi\cos\theta + C\dot\psi$

$\theta = \theta_0 =$ const. , $\dot\phi =$ const.

$\Rightarrow \dot\theta = 0$

$\Rightarrow$ (i) $A\omega_{\perp}\sin(\Omega t+\alpha) = A\dot\phi\sin\theta_0\sin\psi$

(ii) $A\omega_{\perp}\cos(\Omega t+\alpha) = A\dot\phi\sin\theta_0\cos\psi$

(iii) $C\omega_{\parallel} = C(\dot\phi\cos\theta_0 + \dot\psi)$

$\Rightarrow$ (i)/(ii) $\tan(\Omega t+\alpha) = \tan\psi$

$\Rightarrow \boxed{\psi = \Omega t + \alpha}$

in (i):

$\boxed{\omega_{\perp} = \dot\phi \sin\theta_0}$

$\Rightarrow \phi(t) = \frac{\omega_{\perp} t}{\sin\theta_0} + \phi_0$

(111) $\omega_{\parallel} = \dot{\phi}\cos\theta_0 + \dot{\psi} = \frac{\omega_\perp}{\tan\theta_0} + \Omega = \frac{\omega_\perp}{\tan\theta_0} + \frac{A-C}{A}\,\omega_\parallel$

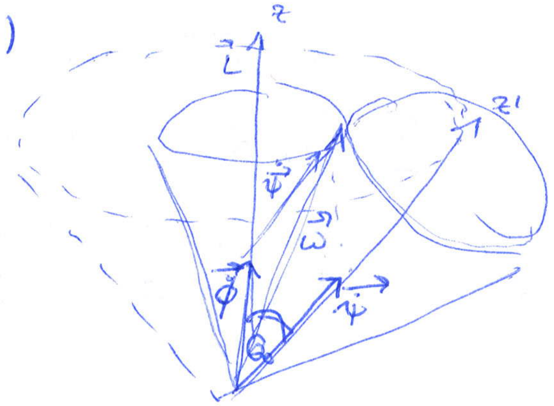
$$\Rightarrow \cancel{\frac{A\omega_\parallel}{A}} = \frac{\omega_\perp}{\tan\theta} + \cancel{\frac{A}{A}\omega_\parallel} - \frac{C}{A}\omega_\parallel \quad \Rightarrow \quad \tan\theta_0 = \frac{A\omega_\perp}{C\omega_\parallel}$$

(↳ integration constants: $\omega_\perp, \alpha, \phi_0, \theta_0$ ... what are the other two?
orientation in LF was chosen to be $\upuparrows \vec{e}_z$)

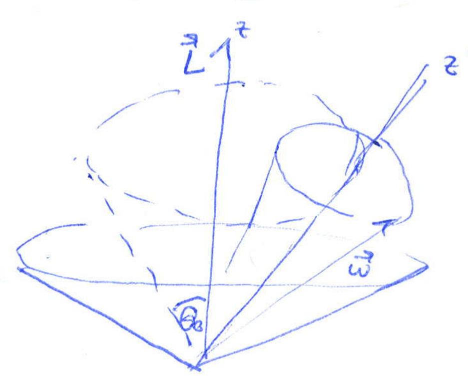
θ: angle between z axis in LF and z' axis in BF
↳ symmetry axis is describing a cone with fixed θ_0 around $(\vec{L})_{LF}$.
with $\dot{\phi}$ (precession)
↳ nutation cone

$\dot{\psi}$: rotation about symmetry axis.

$\vec{\omega} = \vec{\dot{\phi}} + \vec{\dot{\psi}}$



A > C



A < C

Heavy symmetric top

$$T = T_{rot} = \frac{1}{2}A\left(\omega_{x'}^2 + \omega_{y'}^2\right) + \frac{1}{2}C\omega_{z'}^2$$

$$\omega_{x'} = \dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_{y'} = \dot{\phi}\,\sin\theta\,\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_{z'} = \dot{\phi}\,\cos\theta + \dot{\psi}$$

$$\Rightarrow T = \frac{1}{2}A\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$$

$V = Mgl\cos\theta$

$$\Rightarrow \mathcal{L} = \frac{1}{2}A\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgl\cos\theta$$

Note: ϕ, ψ are cyclic!

$$\frac{\partial \mathcal{L}}{\partial \dot\psi} = C(\dot\psi + \dot\phi\cos\theta) = p_\psi = \text{const.}$$

(Note: $p_\psi = C\,\omega_{z'} = L_{z'}$ for a symmetric top but p_ϕ is not...)

$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = (A\sin^2\theta + C\cos^2\theta)\dot\phi + C\dot\psi\cos\theta = A\dot\phi\sin^2\theta + p_\psi\cos\theta \equiv p_\phi = \text{const.}$$

Rearrange:

$$\Rightarrow \quad \dot\psi + \dot\phi\cos\theta = \frac{p_\psi}{C} \qquad \left(= \omega_{z'},\ \text{projection of } \vec{\dot\phi} + \vec{\dot\psi} \text{ on BF } z' \text{ axis}\right)$$

$$\dot\phi = \frac{p_\phi - p_\psi\cos\theta}{A\sin^2\theta}$$

$$E = T + V = \frac{1}{2}(A\dot\theta^2 + A\dot\phi^2\sin^2\theta) + \underbrace{\tfrac{1}{2}C(\dot\psi + \dot\phi\cos\theta)^2}_{=\frac{p_\psi^2}{2C}} + Mgl\cos\theta$$

since $\frac{\partial \mathcal{L}}{\partial t} = 0$

$$\rightsquigarrow \quad E' = E - \frac{p_\psi^2}{2C} = \text{const.}$$

$$= \frac{1}{2}A\dot\theta^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2A\sin^2\theta} + Mgl\cos\theta \quad \leftarrow \text{only a function of } \theta$$

(const.)

$$\Rightarrow \quad V_{eff}(\theta) \equiv \frac{(p_\phi - p_\psi\cos\theta)^2}{2A\sin^2\theta} + Mgl\cos\theta$$

<u>separation of variables</u>

$$\int_{\theta_0}^{\theta(t)} \frac{d\theta'}{\sqrt{E' - V_{eff}(\theta')}} = \sqrt{\frac{2}{A}}\,(t - t_0)$$

$\Rightarrow \theta(t)$, $\Rightarrow \phi(t)$ from $\dot\phi$ $\Rightarrow \psi(t)$ by integrating conservation laws.

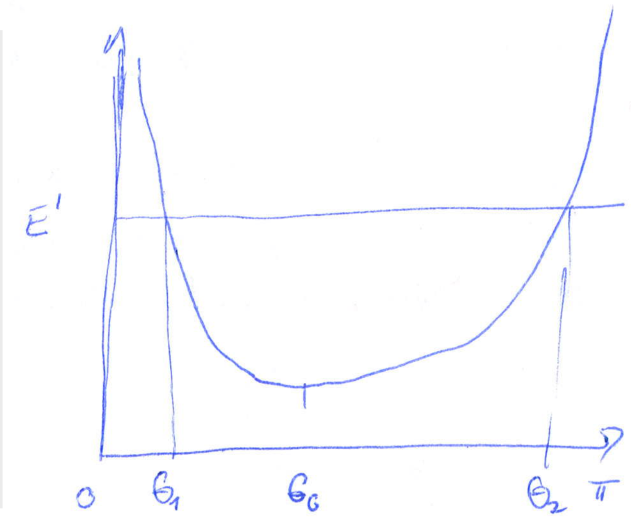
Messy in general, but we can discuss the effective potential

$V_{eff}(\theta) \to \infty$ for $\theta \to 0, \pi$

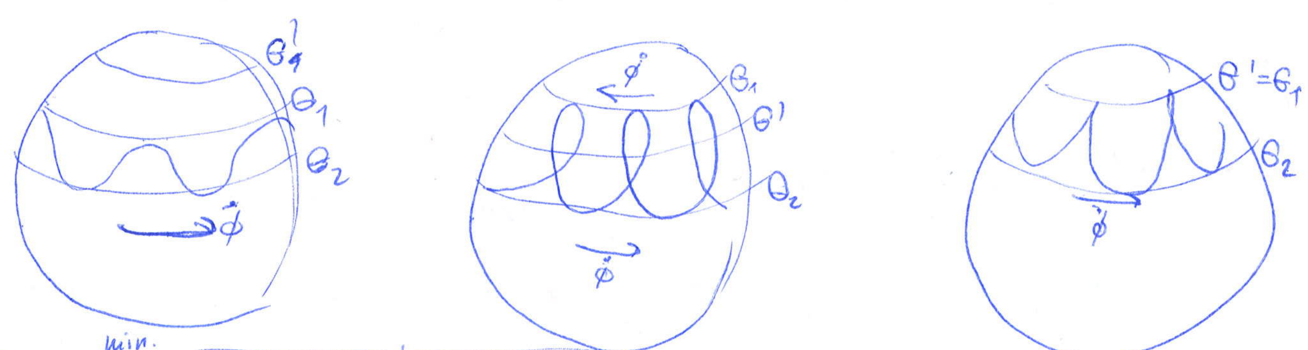
$$V'_{eff}(\theta_0) = -Mgl\sin\theta_0 + \frac{(p_\phi - p_\psi\cos\theta_0)(p_\psi - p_\phi\cos\theta_0)}{A\sin^3\theta_0} = 0$$

$\hookrightarrow$ can show graphically that there is only one minimum.

$V_{eff}(\theta)$



θ_1, θ_2 are turning points for the nutation.



- ~~For~~ At $\theta_0^{\text{min.}} = \theta_1 = \theta_2$ (as in the gravitational case)
- ~~First~~ $V_{eff}(\theta') = Mgl\cos\theta'$

for $\cos\theta' = \frac{P_\phi}{P_\psi}$: $V_{eff}(\theta') = \frac{(P_\phi - P_\psi\cos\theta')^2}{2A\sin^2\theta'} + Mgl\cos\theta'$

$$= Mgl\cos\theta'$$

Note that we also have $\dot\phi = \frac{P_\phi - P_\psi\cos\theta}{A\sin^2\theta}$, so $\dot\phi$ changes sign at θ'!